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Macdonald's Evaluation Conjectures and Difference Fourier Transform

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Introduction

Generalizing the characters of compact simple Lie groups, Ian Macdonald introduced in [M1,M2] and other works remarkable symmetric trigonometric polynomials dependent on the parameters q, t. He came up with four main conjectures formulated for arbitrary root systems. A new approach to the Macdonald theory was suggested in [C1] on the basis of double affine Hecke algebras (new objects in mathematics). In [C2] the norm conjecture (including the famous constant term conjecture [M3]) and the conjecture about the denominators of the coefficients of the Macdonald polynomials were proved. This paper contains the proof of the remaining two (the duality and evaluation conjectures).

The evaluation conjecture (now a theorem) is in fact a q, t-generalization of the classic Weyl dimension formula. One can expect interesting applications of this theorem since the so-called q-dimensions are undoubtedly important. It is likely that we can incorporate the Kac-Moody case as well. The necessary technique was developed in [C4].

As to the duality theorem (in its complete form), it states that the generalized trigonometric-difference zonal Fourier transform is self-dual (at least formally). We define this q, t-transform in terms of double affine Hecke algebras. The most natural way to check the self-duality is to use the connection of these algebras with the so-called elliptic braid groups (the Fourier involution will turn into the transposition of the periods of an elliptic curve).

The classical trigonometric-differential Fourier transform (corresponding to the limit $q = t^k$ as $t \to 1$ for certain special k) plays one of the main roles in the harmonic analysis on symmetric spaces. It sends symmetric trigonometric polynomials to the corresponding radial parts of Laplace operators (Harish-Chandra, Helgason) and is not self-dual. The calculation of its inverse (the Plancherel theorem) is always challenging and involving.

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In the rational-differential setting, Charles Dunkl introduced the generalized Hankel transform which appeared to be self-dual [D,J]. We demonstrate in this paper that one can save this very important property if trigonometric polynomials come together with difference operators. At the moment, it is mostly an algebraic observation (the difference-analitical aspects were not touched upon).

The root systems of type A_n are always rather special. First of all, we note the $q \leftrightarrow t$ symmetry and very interesting positivity conjectures (Macdonald [M1], Garsia, Haiman [GH]). Then the Macdonald polynomials can be interpreted as generalized characters (Etingof, Kirillov [EK1]). The difference Fourier transform also has particular features (we discuss this a little at the end of the paper). By the way, due to Andrews one can add n new parameters q and still the constant term conjecture (proved by Bressoud and Zeilberger) will hold, but there are no related orthogonal polynomials. In the differential setting, the corresponding symmetric polynomials (Jack polynomials) are quite remarkable as well (Hanlon, Stanley).

As to the differential theory, the Macdonald- Mehta conjectures were proved finally by Eric Opdam [O1] (see also [O2]) excluding the duality conjecture which collapses (the Fourier transform is not self-dual!). He used the Heckman-Opdam operators (including the shift operator - see [01,He]). We use their difference counterparts from [C1,C2] defined by means of double affine Hecke algebras. We mention that the latter algebras were not absolutely necessary in [C2] to prove the norm conjecture (the classic affine Hecke algebras are enough). Only in this paper the double Hecke algebras work at their full potential to ensure the duality.

We note that this paper is a part of a new program in the harmonic analysis of symmetric spaces based on certain remarkable representations of Hecke algebras in terms of Dunkl and Demazure operators instead of Lie groups and Lie algebras. It gave already a parametric deformation of the the classical theory (see [O1,He,C5]) directly connected with the so-called quantun many-body problem (Calogero, Sutherland, Moser, Olshanetsky, Perelomov). Then it was extended (in the algebraic context) to the difference, elliptic, and finally to the difference-elliptic case [C4] presumably corresponding to the quantum Kac-Moody algebras. Presumably because the harmonic analysis for the latter algebras does not exist.

The duality-evaluation conjecture. Let $R = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbf{R}^n \ni z, z'$, W the Weyl group generated by the the reflections s_α. We assume that $(\alpha, \alpha) = 2$ for long α. Let us fix the set R_+ of positive roots ($R_- = -R_+$), the corresponding simple roots $\alpha_1, ..., \alpha_n$, and their dual counterparts $a_1, ..., a_n, a_i =$

$\alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. The dual fundamental weights $b_1, ..., b_n$ are determined from the relations $(b_i, \alpha_j) = \delta_i^j$ for the Kronecker delta. We will also introduce the dual root system $R^\vee = \{\alpha^\vee, \alpha \in R\}, R_+^\vee$, and the lattices

$$A = \oplus_{i=1}^n \mathbf{Z} a_i \subset B = \oplus_{i=1}^n \mathbf{Z} b_i,$$

$A_\pm, B_\pm$ for $\mathbf{Z}_\pm = \{m \in \mathbf{Z}, \pm m \geq 0\}$ instead of $\mathbf{Z}$. (In the standard notations, $A = Q^\vee,\ B = P^\vee$ - see [B].) Later on,

$$\begin{aligned} \nu_\alpha = \nu_{\alpha^\vee} \ &= \ (\alpha,\alpha),\ \nu_i \ = \ \nu_{\alpha_i},\ \nu_R \ = \{\nu_\alpha, \alpha \in R\}, \\ \rho_\nu \ &= \ (1/2) \sum_{\nu_\alpha = \nu} \alpha \ = \ (\nu/2) \sum_{\nu_i = \nu} b_i, \quad \text{for} \quad \alpha \in R_+, \\ r_\nu \ &= \ \rho_\nu^\vee \ = \ (2/\nu)\rho_\nu \ = \ \sum_{\nu_i = \nu} b_i, \quad 2/\nu = 1,2,3. \end{aligned} \tag{0.1}$$

Let us put formally $x_i = exp(b_i),\ x_b = exp(b) = \prod_{i=1}^n x_i^{k_i}$ for $b = \sum_{i=1}^n k_i b_i$, and introduce the algebra $\mathbf{C}(\delta,q)[x]$ of polynomials in terms of $x_i^{\pm 1}$ with the coefficients belonging to the field $\mathbf{C}(\delta,q)$ of rational functions in terms of indefinite complex parameters $\delta, q_\nu, \nu \in \nu_R$ (we will put $q_\alpha = q_{\nu_\alpha} = q_{\alpha^\vee}$). The coefficient of $x^0 = 1$ (*the constant term*) will be denoted by $\langle\ \rangle$. The following product is a Laurent series in x with the coefficients in $\mathbf{C}(\delta,q)$:

$$\mu \ = \ \prod_{a \in R_+^\vee} \prod_{i=0}^\infty \frac{(1 - x_a \delta_a^i)(1 - x_a^{-1}\delta_a^{i+1})}{(1 - x_a q_a \delta_a^i)(1 - x_a^{-1} q_a^{-1} \delta_a^{i+1})}, \tag{0.2}$$

where $\delta_a = \delta_\nu = \delta^{2/\nu}$ for $\nu = \nu_a$. We note that $\mu \in \mathbf{C}(\delta,q)[x]$ if $q_\nu = \delta_\nu^{k_\nu}$ for $k_\nu \in \mathbf{Z}_+$.

The *monomial symmetric polynomials* $m_b = \sum_{c \in W(b)} x_c$ for $b \in B_-$ form a base of the space $\mathbf{C}[x]^W$ of all W-invariant polynomials. Setting $\bar{x}_b \stackrel{def}{=} x_{-b}$,

$$\langle f, g \rangle \ = \langle \mu f \ \bar{g} \rangle \quad \text{for} \quad f, g \in \mathbf{C}(\delta,q)[x]^W, \tag{0.3}$$

we introduce the *Macdonald polynomials* $p_b(x),\ b \in B_-$, by means of the conditions

$$\begin{aligned} p_b - m_b \ \in \ \oplus_c \mathbf{C}(\delta,q) m_c,\ \langle p_b, m_c \rangle = 0, \\ \text{where} \ \ c \in B_-,\ c - b \in A_+, c \neq b. \end{aligned} \tag{0.4}$$

They can be determined by the Gram - Schmidt process because the pairing (see [M1,M2]) is non-degenerate and form a basis in $\mathbf{C}(\delta,q)[x]^W$. Let $x_i(q^{-\rho}\delta^b) = \delta^{(b,b_i)} \prod_\nu q_\nu^{-(b_i,\rho_\nu)}$.

Main Theorem. *Given $b, c \in B_-$ and the corresponding Macdonald polynomials p_b, p_c,*

$$p_b(q^{-\rho}\delta^c)p_c(q^{-\rho}) \;=\; p_c(q^{-\rho}\delta^b)p_b(q^{-\rho}), \tag{0.5}$$

$$p_b(q^{-\rho}) \;=\; \prod_\nu q_\nu^{(\rho_\nu, b)} \prod_{a\in R_+^\vee, 0\le j<\infty} \Big(\frac{(1-\delta_a^{j-(b,a^\vee)}\prod_\nu q_\nu^{(\rho_\nu,a)})(1-q_a\delta_a^j\prod_\nu q_\nu^{(\rho_\nu,a)})}{(1-q_a\delta_a^{j-(b,a^\vee)}\prod_\nu q_\nu^{(\rho_\nu,a)})(1-\delta_a^j\prod_\nu q_\nu^{(\rho_\nu,a)})}\Big). \tag{0.6}$$

The right hand side of (0.6) is a rational function in terms of δ, q (we used $a^\vee = 2a/(a,a)$ to make it more transparent). We mention that there is a straightforward passage to the case where μ is introduced for $\alpha \in R_+$ instead of $a \in R_+^\vee$ (see [C2]) and to non-reduced root systems.

The second formula was conjectured by Macdonald (see (12.10),[M2]). He also formulated an equivalent version of (0.5) in one of his lectures (1991). Both statements seem to be esatblished in 1988 by Koornwinder for A_n (his proof was not published) and by Macdonald (to be published). Recently the paper by Etingof and Kirillov [EK2] appeared were they use their interpretation of the Macdonald polynomials to check the above theorem (and the norm conjecture) in the case of A_n. As to other root systems, it seems that almost nothing was known (excluding BC_1 and certain special values of the parameters).

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1. Double affine Hecke algebras

The vectors $\tilde\alpha = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the *affine root system* $R^a \supset R$ ($z \in \mathbf{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{def}{=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta \in R$. The corresponding set R_+^a of positive roots coincides with $R_+ \cup \{[\alpha, k],\ \alpha \in R,\ k > 0\}$.

We denote the Dynkin diagram and its affine completion with $\{\alpha_j, 0 \le j \le n\}$ as the vertices by Γ and Γ^a. Let $m_{ij} = 2, 3, 4, 6$ if α_i and α_j are joined by 0,1,2,3 laces respectively. The set of the indices of the images of α_0 by all the automorphisms of Γ^a will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = r \in O, r \ne 0$. The elements b_r for $r \in O^*$ are the so-called minuscule weights ($(b_r, \alpha) \le 1$ for $\alpha \in R_+$).

Given $\tilde\alpha = [\alpha, k] \in R^a,\ b \in B$, let

$$s_{\tilde\alpha}(\tilde z) \;=\; \tilde z - (z, \alpha^\vee)\tilde\alpha, \quad b'(\tilde z) \;=\; [z, \zeta - (z, b)] \tag{1.1}$$

for $\tilde z = [z, \zeta] \in \mathbf{R}^{n+1}$.

The *affine Weyl group* W^a is generated by all $s_{\tilde\alpha}$ (we write $W^a =< s_{\tilde\alpha}, \tilde\alpha \in R_+^a >$). One can take the simple reflections $s_j = s_{\alpha_j}, 0 \le j \le n$, as its

generators and introduce the corresponding notion of the length. This group is the semi-direct product $W\ltimes A'$ of its subgroups $W =< s_\alpha, \alpha\in R_+ >$ and $A'=\{a', a\in A\}$, where

$$a' = \ s_\alpha s_{[\alpha,1]} = \ s_{[-\alpha,1]}s_\alpha \ \text{ for } \ a=\alpha^\vee,\ \alpha\in R. \tag{1.2}$$

The *extended Weyl group* W^b generated by W and B' (instead of A') is isomorphic to $W\ltimes B'$:

$$(wb')([z,\zeta]) \ = \ [w(z),\zeta-(z,b)] \ \text{ for } \ w\in W, b\in B. \tag{1.3}$$

Given $b_+\in B_+$, let

$$\omega_{b_+} = w_0 w_0^+ \in W,\ \pi_{b_+} = b'_+(\omega_{b_+})^{-1} \ \in \ W^b,\ \omega_i=\omega_{b_i}, \pi_i=\pi_{b_i}, \tag{1.4}$$

where w_0 (respectively, w_0^+) is the longest element in W (respectively, in W_{b_+} generated by s_i preserving b_+) relative to the set of generators $\{s_i\}$ for $i>0$.

We will use here only the elements $\pi_r = \pi_{b_r}, r\in O$. They leave Γ^a invariant and form a group denoted by Π, which is isomorphic to B/A by the natural projection $\{b_r\to\pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta,\alpha_i, i>0\}$. The relations $\pi_r(\alpha_0)=\alpha_r=(\omega_r)^{-1}(-\theta)$ distinguish the indices $r\in O^*$. Moreover (see e.g. [C2]):

$$W^b=\Pi\ltimes W^a, \ \text{ where } \ \pi_r s_i \pi_r^{-1} = s_j \ \text{ if } \ \pi_r(\alpha_i)=\alpha_j,\ 0\le j\le n. \tag{1.5}$$

We extend the notion of the length to W^b. Given $\nu\in\nu_R,\ r\in O^*,\ \tilde w\in W^a$, and a reduced decomposition $\tilde w \ = \ s_{j_l}...s_{j_2}s_{j_1}$ with respect to $\{s_j, 0\le j\le n\}$, we call $l \ = \ l(\hat w)$ the *length* of $\hat w=\pi_r\tilde w\in W^b$. Setting

$$\begin{aligned}\lambda(\hat w) =& \{\tilde\alpha^1=\alpha_{j_1},\ \tilde\alpha^2=s_{j_1}(\alpha_{j_2}),\ \tilde\alpha^3=s_{j_1}s_{j_2}(\alpha_{j_3}),\ldots\\ &\ldots,\tilde\alpha^l=\tilde w^{-1}s_{j_l}(\alpha_{j_l})\},\end{aligned} \tag{1.6}$$

one can represent

$$\begin{aligned} l=|\lambda(\hat w)| &= \sum_\nu l_\nu, \quad \text{for } \ l_\nu=l_\nu(\hat w)=|\lambda_\nu(\hat w)|,\\ \lambda_\nu(\hat w) &= \{\tilde\alpha^m,\ \nu(\tilde\alpha^m)=\nu(\tilde\alpha_{j_m})=\nu\}, 1\le m\le l,\end{aligned} \tag{1.7}$$

where $|\ |$ denotes the number of elements, $\nu([\alpha,k]) \stackrel{def}{=} \nu_\alpha$.

For instance,

$$\begin{aligned} l_\nu(b') \ &= \ \sum_\alpha |(b,\alpha)|,\ \alpha\in R_+, \nu_\alpha=\nu\in\nu_R,\\ l_\nu(b'_+) \ &= \ 2(b_+,\rho_\nu) \ \text{ when } \ b_+\in B_+.\end{aligned} \tag{1.8}$$

Here $|\ |$ = absolute value. Later on b and b' will not be distinguished.

We put $m=2$ for D_{2k} and C_{2k+1}, $m=1$ for C_{2k}, B_k, otherwise $m=|\Pi|$. The definition involves the parameters $\delta, \{q_\nu, \nu \in \nu_R\}$ and independent variables $X_1, \ldots, X_n$. Let us set

$$
\begin{aligned}
& q_{\tilde\alpha} = q_{\nu(\tilde\alpha)},\ q_j = q_{\alpha_j}, \quad \text{where } \ \tilde\alpha \in R^a, 0 \le j \le n, \\
& X_{\tilde b} \ = \ \prod_{i=1}^n X_i^{k_i} \delta^k \ \text{ if } \ \tilde b = [b,k], \\
& \quad \text{for } \ b = \sum_{i=1}^n k_i b_i \in B,\ k \in \frac{1}{m}\mathbf{Z}.
\end{aligned} \tag{1.9}
$$

Later on $\mathbf{C}_\delta$ is the field of rational functions in $\delta^{1/m}$, $\mathbf{C}_\delta[X] = \mathbf{C}_\delta[X_b]$ means the algebra of polynomials in terms of $X_i^{\pm 1}$ with the coefficients depending on $\delta^{1/m}$ rationally. We replace $\mathbf{C}_\delta$ by $\mathbf{C}_{\delta,q}$ if the functions (coefficients) also depend rationally on $\{q_\nu^{1/2}\}$.

Let $([a,k],[b,l]) = (a,b)$ for $a,b \in B$, $[\alpha,k]^\vee = [\alpha^\vee, k]$, $a_0 = \alpha_0$, $\nu_{\alpha^\vee} = \nu_\alpha$, and $\alpha_{r^*} \stackrel{def}{=} \pi_r^{-1}(\alpha_0)$ for $r \in O^*$.

Definition 1.1. *The double affine Hecke algebra* $\mathfrak{H}$ *(see [C1,C2]) is generated over the field* $\mathbf{C}_{\delta,q}$ *by the elements* $\{T_j,\ 0 \le j \le n\}$*, pairwise commutative* $\{X_b,\ b \in B\}$ *satisfying (1.9), and the group* Π *where the following relations are imposed:*

(o) $(T_j - q_j^{1/2})(T_j + q_j^{-1/2}) \ = \ 0,\ 0 \ \le \ j \ \le \ n;$
(i) $T_iT_jT_i\ldots \ = \ T_jT_iT_j\ldots,$ *m_{ij} factors on each side;*
(ii) $\pi_r T_i \pi_r^{-1} \ = \ T_j$ *if* $\pi_r(\alpha_i) = \alpha_j;$
(iii) $T_i X_b T_i \ = \ X_b X_{a_i}^{-1}$ *if* $(b,\alpha_i) = 1,\ 1 \le i \le n;$
(iv) $T_0 X_b T_0 \ = \ X_{s_0(b)} \ = \ X_b X_\theta \delta^{-1}$ *if* $(b,\theta) = -1;$
(v) $T_i X_b \ = \ X_b T_i$ *if* $(b,\alpha_i) = 0$ *for* $0 \le i \le n;$
(vi) $\pi_r X_b \pi_r^{-1} \ = \ X_{\pi_r(b)} \ = \ X_{\omega_r^{-1}(b)} \delta^{(b_{r^*}, b)},\ r \in O^*.$

□

Given $\tilde w \in W^a, r \in O,$ the product

$$
T_{\pi_r \tilde w} \stackrel{def}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where } \ \tilde w = \prod_{k=1}^l s_{i_k}, l = l(\tilde w), \tag{1.10}
$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$
T_{\hat v} T_{\hat w} \ = \ T_{\hat v \hat w} \ \text{ whenever } l(\hat v \hat w) = l(\hat v) + l(\hat w) \ \text{ for } \ \hat v, \hat w \in W^b. \tag{1.11}
$$

In particular, we arrive at the pairwise commutative elements

$$
Y_b \ = \ \prod_{i=1}^n Y_i^{k_i} \ \text{ if } \ b = \sum_{i=1}^n k_i b_i \in B, \quad \text{where } \ Y_i \stackrel{def}{=} T_{b_i}, \tag{1.12}
$$

satisfying the relations

$$
\begin{aligned}
T_i^{-1}Y_bT_i^{-1} \ &= \ Y_bY_{a_i}^{-1} \ \text{ if } \ (b,\alpha_i)=1,\\
T_iY_b \ &= \ Y_bT_i \ \text{ if } \ (b,\alpha_i)=0,\ 1\le i\le n.
\end{aligned} \tag{1.13}
$$

Let us introduce the following elements from $\mathbf{C}_q^n$:

$$
\begin{aligned}
&q^{\pm\rho} \stackrel{def}{=} (l_q(b_1)^{\pm 1},\dots,l_q(b_n)^{\pm 1}), \quad \text{where}\\
&l_q(\hat{w}) \stackrel{def}{=} \prod_{\nu\in\nu_R} q_\nu^{l_\nu(\hat{w})/2},\ \hat{w}\in W^b,
\end{aligned} \tag{1.14}
$$

and the corresponding *evaluation maps*:

$$
X_i(q^{\pm\rho}) = l_q(b_i)^{\pm 1} = Y_i(q^{\pm\rho}),\ 1\le i\le n. \tag{1.15}
$$

For instance, $X_{a_i}(q^\rho) \ = \ l_q(a_i)=q_i$ (see (1.8)).

THEOREM 1.2. *i) The elements $H\in\mathfrak{H}$ have the unique decompositions*

$$
H=\sum_{w\in W} g_wT_wf_w,\ g_w\in\mathbf{C}_{\delta,q}[X],\ f_w\in\mathbf{C}_{\delta,q}[Y]. \tag{1.16}
$$

ii) The map

$$
\begin{aligned}
\varphi :&X_i\to Y_i^{-1},\quad Y_i\to X_i^{-1},\quad T_i\to T_i,\\
&q_\nu\to q_\nu,\ \delta\to\delta,\ \nu\in\nu_R,\ 1\le i\le n.
\end{aligned} \tag{1.17}
$$

can be extended to an anti-involution ($\varphi(AB)=\varphi(B)\varphi(A)$) of $\mathfrak{H}$.

iii) The linear functional on $\mathfrak{H}$

$$
[\![\sum_{w\in W} g_wT_wf_w]\!] \ = \ \sum_{w\in W} g_w(q^{-\rho})l_q(w)f_w(q^\rho) \tag{1.18}
$$

is invariant with respect to φ. The bilinear form

$$
[\![F,G]\!] \stackrel{def}{=} [\![\varphi(G)H]\!],\ G,H\in\mathfrak{H}\ , \tag{1.19}
$$

is symmetric ($[\![G,H]\!]=[\![H,G]\!]$) and non-degenerate.

Proof. The first statement is from Theorem 2.3 [C2]. The map φ is the composition of the involution (see [C1])

$$
\begin{aligned}
&X_i\to Y_i,\quad Y_i\to X_i,\quad T_i\to T_i^{-1},\\
&q_\nu\to q_\nu^{-1},\ \delta\to\delta^{-1},\ 1\le i\le n,
\end{aligned} \tag{1.20}
$$

and the main anti-involution "*" from [C2], sending

$$
\begin{aligned}
&X_i\to X_i^{-1},\quad Y_i\to Y_i^{-1},\quad T_i\to T_i^{-1},\\
&q_\nu\to q_\nu^{-1},\ \delta\to\delta^{-1},\ 0\le i\le n.
\end{aligned} \tag{1.21}
$$

The other claims follow directly from the definition of $[\![\ ]\!]$. □

One can extend $[\![\ ]\!]$ to the localization of $\mathfrak{H}$ with respect to all polynomials in X (or in Y). The algebra becomes the semi-direct product of $\mathbf{C}[W^b]$ and $\mathbf{C}(X)$ after this (see [C3]). Sometimes it is also convenient to involve proper completions of $\mathbf{C}(X)$ (see the end of the paper).

2. Difference operators

Setting (see the Introduction)

$$x_{\tilde{b}} = \prod_{i=1}^{n} x_i^{k_i} \delta^k \ \text{ if } \ \tilde{b} = [b, k], b = \sum_{i=1}^{n} k_i b_i \in B, \ k \in \frac{1}{m}\mathbf{Z}, \tag{2.1}$$

for independent $x_1, \dots, x_n$, we will consider $\{X\}$ as operators acting in $\mathbf{C}_\delta[x] = \mathbf{C}_\delta[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$:

$$X_{\tilde{b}}(p(x)) \ = \ x_{\tilde{b}} p(x), \ p(x) \in \mathbf{C}_\delta[x]. \tag{2.2}$$

The elements $\hat{w} \in W^b$ act in $\mathbf{C}_\delta[x]$ by the formulas:

$$\hat{w}(x_{\tilde{b}}) \ = \ x_{\hat{w}(\tilde{b})}. \tag{2.3}$$

In particular:

$$\pi_r(x_b) \ = \ x_{\omega_r^{-1}(b)} \delta^{(b_{r^*}, b)} \ \text{ for } \ \alpha_{r^*} \ = \ \pi_r^{-1}(\alpha_0), \ r \in O^*. \tag{2.4}$$

The *Demazure-Lusztig operators* (see [KL, KK, C1], and [C2] for more detail)

$$\hat{T}_j \ = \ q_j^{1/2} s_j \ + \ (q_j^{1/2} - q_j^{-1/2})(X_{a_j} - 1)^{-1}(s_j - 1), \ 0 \le j \le n. \tag{2.5}$$

act in $\mathbf{C}_{\delta,q}[x]$ naturally. We note that only $\hat{T}_0$ depends on δ:

$$\begin{aligned} \hat{T}_0 \ &= q_0^{1/2} s_0 \ + \ (q_0^{1/2} - q_0^{-1/2})(\delta X_\theta^{-1} - 1)^{-1}(s_0 - 1), \\ &\text{where } \ s_0(X_i) \ = \ X_i X_\theta^{-(b_i, \theta)} \delta^{(b_i, \theta)}. \end{aligned} \tag{2.6}$$

THEOREM 2.1. *The map $T_j \to \hat{T}_j, \ X_b \to X_b$ (see (1.9,2.2)), $\pi_r \to \pi_r$ (see (2.4)) induces a $\mathbf{C}_{\delta,q}$-linear homomorphism from $\mathfrak{H}$ to the algebra of linear endomorphisms of $\mathbf{C}_{\delta,q}[x]$. This representation is faithful and remains faithful when δ, q take any non-zero values assuming that δ is not a root of unity (see [C2]). The image $\hat{H}$ is uniquely determined from the following condition:*

$$\begin{aligned} &\hat{H}(f(x)) \ = \ g(x) \ \text{ for } \ H \in \mathfrak{H}\ , \ \text{ if } \ Hf(X) \ = \\ &g(X) + \sum_{i=0}^{n} H_i(T_i - q_i) + \sum_{r \in O^*} H_r(\pi_r - 1), \ \text{ where } \ H_i, H_r \in \mathfrak{H}\ . \end{aligned} \tag{2.7}$$

□

Due to Theorem 1.2, an arbitrary $H \in \mathfrak{H}$ can be uniquely represented in the form

$$
\begin{aligned}
H &= \sum_{b\in B, w\in W} g_{b,w} Y_b T_w,\ g_{b,w} \in \mathbf{C}_{\delta,q}[X], \\
&= \sum_{b\in B, w\in W} T_w X_b g'_{b,w},\ g'_{b,w} \in \mathbf{C}_{\delta,q}[Y].
\end{aligned} \tag{2.8}
$$

We set:

$$
\begin{aligned}
[H]_\dagger &= \sum_{b\in B, w\in W} g_{b,w} Y_b l_q(w),\ {}_\dagger[H] = \sum_{b\in B, w\in W} l_q(w) X_b g_{b,w}, \\
[H]_\ddagger &= \sum_{b\in B, w\in W} g_{b,w} [\![Y_b T_w]\!],\ {}_\ddagger[H] = \sum_{b\in B, w\in W} [\![T_w X_b]\!] g'_{b,w}.
\end{aligned} \tag{2.9}
$$

One easily checks that

$$
\begin{aligned}
[\![H_1 H_2]\!] &= [\![H_1 [H_2]_\dagger]\!] = [\![{}_\dagger[H_1] H_2]\!] = \\
&\quad [\![H_1 [H_2]_\ddagger]\!] = [\![{}_\ddagger[H_1] H_2]\!] \ \text{ for } \ H_1, H_2 \in \mathfrak{H}\ .
\end{aligned} \tag{2.10}
$$

Let us represent the image $\hat{H}$ of H as follows:

$$
\hat{H} = \sum_{b\in B, w\in W} h_{b,w} bw, = \sum_{b\in B, w\in W} wbh'_{b,w}. \tag{2.11}
$$

where $h_{b,w}, h'_{b,w}$ belong to the field $\mathbf{C}_{\delta,q}(X)$ of rational functions in $X_1, ..., X_n$. We extend the above operations to arbitrary operators in the form (2.11):

$$
[\hat{H}]_\dagger = \sum h_{b,w} b,\ {}_\dagger[\hat{H}] = \sum b h'_{b,w}\ ,\ [\![\hat{H}]\!] = \sum h_{b,w}(q^{-\rho}). \tag{2.12}
$$

These operations commute with the homomorphism $H \to \hat{H}$.

Let us define the *difference Harish-Chandra map* (see [C2], Proposition 3.1):

$$
\chi\Big(\sum_{w\in W, b\in B} h_{b,w} bw \Big) = \sum_{b\in B, w\in B} h_{b,w}(\diamondsuit) y_b \ \in \mathbf{C}_{\delta,q}[y], \tag{2.13}
$$

where $\diamondsuit \stackrel{def}{=} (X_1 = ... = X_n = 0)$, $\{y_b\}$ is one more set of variables introduced for independent $y_1, ..., y_n$ in the same way as $\{x_b\}$ were.

Proposition 2.2. *Setting*

$$
\mathcal{L}_f = f(Y),\ \hat{\mathcal{L}}_f = f(\hat{Y}),\ L_f = L_f^{\delta,q} \stackrel{def}{=} [(\hat{\mathcal{L}}_f)]_\dagger \tag{2.14}
$$

for $f = \sum_b g_b y_b \in \mathbf{C}_{\delta,q}[y]$, *one has:*

$$
\chi(\hat{\mathcal{L}}_f) = \chi(L_f) = [\![f(Y)]\!] = \sum_{b\in B} g_b \prod_\nu q_\nu^{(b,\rho_\nu)} y_b. \tag{2.15}
$$

□

The proof of the following theorem repeats the proof of Theorem 4.5,[C2] (where the relations $q_\nu = \delta_\nu^{k_\nu}$ for $k_\nu \in \mathbf{Z}_+$ were imposed). We note that once (2.16) is known for these special q it holds true for all δ, q since all the coefficients of difference operators and polynomials are rational in δ, q.

THEOREM 2.3. *The difference operators* $\{L_f,\ f(y_1,\dots,y_n)\in \mathbf{C}_{\delta,q}[y]^W\}$ *are pairwise commutative,* W*-invariant (i.e* $wL_fw^{-1} = L_f$ *for all* $w\in W$*) and preserve* $\mathbf{C}_{\delta,q}[x]^W$*. The Macdonald polynomials* $p_b = p_b^{\delta,q}(b\in B_-)$ *from (0.4) are their eigenvectors:*

$$L_f(p_b^{\delta,q}) = f(q^\rho\delta^{-b})p_b^{\delta,q},\ \ y_i(q^\rho\delta^{-b}) \overset{def}{=} \delta^{-(b_i,b)}\prod_\nu q_\nu^{(b_i,\rho_\nu)}. \tag{2.16}$$

□

We fix a subset $v\in\nu_R$ and introduce the *shift operator* by the formula

$$\mathcal{G}_v \ = \ (\mathcal{X}_v)^{-1}\mathcal{Y}_v,\ G_v^{\delta,q} \ = \ [\hat{\mathcal{G}}_v]_\dagger \ = \ (\mathcal{X}_v)^{-1}[\hat{\mathcal{Y}}_v]_\dagger, \tag{2.17}$$

$$\mathcal{X}_v = \prod_{\nu_a\in v}((q_aX_a)^{1/2}-(q_aX_a)^{-1/2}),\ \mathcal{Y}_v = \prod_{\nu_a\in v}(q_aY_a^{-1})^{1/2}-(q_aY_a^{-1})^{-1/2}).$$

Here $a=\alpha^\vee\in R_+^\vee, \nu_a=\nu_\alpha, q_a=q_\alpha$, the elements $\mathcal{X}_v=\mathcal{X}_v^q, \mathcal{Y}_v=\mathcal{Y}_v^q$ belong to $\mathbf{C}_q[X], \mathbf{C}_q[Y]$ respectively.

THEOREM 2.4. *The operators* $\hat{\mathcal{G}}_v$ *and* $G_v^{\delta,q}$ *are* W*-invariant and preserve* $\mathbf{C}_{\delta,q}[x]^W$ *(their restrictions to the latter space coincide). Moreover, if* $q_\nu = 1$ *when* $\nu\notin v$ *then*

$$\begin{aligned} G_v^{\delta,q}L_f^{\delta,q} \ &= \ L_f^{\delta,q\delta_v}G_v^{\delta,q} \ \ \textit{for}\ \ f\in\mathbf{C}_{\delta,q}[y]^W,\\ G_v^{\delta,q}(p_b^{\delta,q}) &= g_v^{\delta,q}(b)p_{b+r_v}^{\delta,q\delta_v}, \ \ \textit{for}\\ g_v^{\delta,q}(b) \ &= \prod_{a\in R_+^\vee,\nu_a\in v}(y_a(q^{\rho/2}\delta^{-b/2})-q_ay_a(q^{-\rho/2}\delta^{+b/2})), \end{aligned} \tag{2.18}$$

where $r_v=\sum_{\nu\in v}r_\nu,\ q\delta_v=\{q_\nu\delta^{2/\nu}, q_{\nu'}\}$ *for* $\nu\in v\not\ni\nu'$, $p_c=0$ *for* $c\notin B_-$.

Proof. When $q_\nu=\delta^{2k_\nu/\nu}$ for $k_\nu\in\mathbf{Z}_+$ these statements are in fact from [C2]. They give (2.18) for all δ,q. Indeed, it can be rewritten as follows:

$$[\hat{\mathcal{L}}_f^{\delta,q}\mathcal{X}_v^q]_\dagger \ = \ \mathcal{X}_v^qL_f^{\delta,q\delta_v}, \tag{2.19}$$

where the coefficients of the difference operators on both sides are from $\mathbf{C}_{\delta,q}[X]$. Here we used that $[\mathcal{LM}]_\dagger = [\mathcal{L}]_\dagger[\mathcal{M}]_\dagger$ for arbitrary operators $\mathcal{L},\mathcal{M}$ in the form (2.11) if the second is W-invariant. The remaining formulas can be deduced

from [C2] in the same way (they mean certain identities in $\mathbf{C}_{\delta,q}$ which are enough to check for $q_\nu = \delta^{2k_\nu/\nu}$). One can use (2.15) as well. $\square$

3. Duality and evaluation conjectures

First of all we will use Theorem 1.2 to define the *zonal Fourier transform.* We will sometimes identify the elements $H \in \mathfrak{H}$ with their images $\hat{H}$. The following pairing on $f, g \in \mathbf{C}_{\delta,q}[x]$ is symmetric and non-degenerate:

$$\begin{aligned} &[\![f, g]\!] = [\![f(X), g(X)]\!] = [\![\varphi(f(X))g(X)]\!] = \\ &[\![\bar{f}(Y)g(X)]\!] = \{\mathcal{L}_{\bar{f}}(g(x))\}(q^{-\rho}). \end{aligned} \tag{3.1}$$

Here $\bar{x}_b = x_{-b} = x_b^{-1}$, $\mathcal{L}$ is from (2.14), and we used the main defining property (2.7) of the representation from Theorem 2.1. The pairing remains non-degenerate when restricted to W-invariant polynomials.

Definition 3.1. *The Fourier transforms $\varphi(\mathcal{L}), \varphi(L)$ of $\mathbf{C}_{\delta,q}$-linear operators acting respectively either in $\mathbf{C}_{\delta,q}[x]$ or in $\mathbf{C}_{\delta,q}[x]^W$ are defined from the relations:*

$$\begin{aligned} [\![\mathcal{L}(f), g]\!] &= [\![f, \varphi(\mathcal{L})(g)]\!], \ f, g \in C_{\delta,q}[x], \\ [\![L(f), g]\!] &= [\![f, \varphi(L)(g)]\!], \ f, g \in \mathbf{C}_{\delta,q}[x]^W. \end{aligned} \tag{3.2}$$

If $\mathcal{L}$ preserves $\mathbf{C}_{\delta,q}[x]^W$ then so does $\varphi(\mathcal{L})$ and $\varphi(L) = [\varphi(\mathcal{L})]_\dagger$, where $L = [\mathcal{L}]_\dagger$ is the restriction of $\mathcal{L}$ to the invariant polynomials.

$\square$

This involution ($\varphi^2 = \mathrm{id}$) extends φ from (1.17) by construction. If $f \in C_{\delta,q}[x]^W$, then $\varphi(L_f) = [\bar{f}(X)]_\dagger$. We arrive at the following theorem:

Duality Theorem 3.2. *Given $b, c \in B_-$ and the corresponding Macdonald's polynomials p_b, p_c,*

$$p_b(q^{-\rho}\delta^c)p_c(q^{-\rho}) \ = [\![p_b, p_c]\!] \ = \ [\![p_c, p_b]\!] \ = \ p_c(q^{-\rho}\delta^b)p_b(q^{-\rho}). \tag{3.3}$$

$\square$

To complete this theorem we need to calculate $p_b(q^{-\rho})$. The main step is the formula for $p'((q\delta_v)^{-\rho})$ in terms of $p(q^{-\rho})$, where (see (2.18))

$$p = p_b, \ p' \ = \ p^{q\delta_v}_{b+r_v}, \ p' = (g^q_v(b))^{-1}G^q_v(p).$$

Here and in similar formulas we show the dependence on q omitting δ since the latter will be the same for all polynomials and operators. Let

$$\bar{\mathcal{Y}}^q_v = \prod_{a \in R^\vee_+, \nu_a \in v} ((q_a Y_a)^{1/2} - (q_a Y_a)^{-1/2}).$$

Key Lemma 3.3.

$$
\begin{aligned}
&d^q_v p'((q\delta_v)^{-\rho}) \ = \\
&\prod_{a\in R^\vee_+,\nu_a\in v}\Big(q_a^{-1}y_a(q^{-\rho/2}\delta^{+b/2}) - y_a(q^{+\rho/2}\delta^{-b/2})\Big)p(q^{-\rho}),
\end{aligned}\tag{3.4}
$$

$$
d^q_v \ = \prod_{a\in R^\vee_+,\nu_a\in v}\Big(q_a^{-1}y_a((q\delta_v)^{-\rho/2}) - y_a((q\delta_v)^{+\rho/2})\Big)m_{-r_v}(q^{-\rho}).
$$

Proof. Let us use formula (2.19):

$$
\begin{aligned}
&[\![(\mathcal{Y}^q_v\mathcal{X}^q_v)(\mathcal{X}^q_v)^{-1}\mathcal{L}^q_{\bar{p}'}\mathcal{X}^q_v]\!] \ = \ [\![(\mathcal{Y}^q_v\mathcal{X}^q_v)[(\mathcal{X}^q_v)^{-1}\mathcal{L}^q_{\bar{p}'}\mathcal{X}^q_v]_\dagger]\!] \ = \\
&[\![(\hat{\mathcal{Y}}^q_v\hat{\mathcal{X}}^q_v)\bar{p}'(\hat{Y}^{q\delta_v})]\!] \ = \ [\![(\mathcal{Y}^q_v\mathcal{X}^q_v)]\!]p'((q\delta_v)^{-\rho}).
\end{aligned}\tag{3.5}
$$

On the other hand, it equals:

$$
\begin{aligned}
&[\![\mathcal{Y}^q_v p'(Y)\mathcal{X}^q_v]\!] \ = [\![\mathcal{Y}^q_v p'(X^q)\mathcal{X}^q_v]\!] \ = \\
&[\![\mathcal{Y}^q_v(\mathcal{X}^q_v p'(x))]\!] \ = \pm[\![\bar{\mathcal{Y}}^q_v(\mathcal{X}^q_v p'(x))]\!].
\end{aligned}\tag{3.6}
$$

Here we applied the anti-involution φ ($\varphi(\mathcal{X}) = \mathcal{Y}$, $\varphi(\mathcal{Y}) = \mathcal{X}$), then went from the abstract $[\![\ ]\!]$ to that from (2.12), and used Theorem 2.1. The last transformation requires special comment. We will justify it in a moment.

After this, one can use (2.16):

$$
\begin{aligned}
&[\![\bar{\mathcal{Y}}^q_v(\mathcal{X}^q_v p'(x))]\!] \ = [\![(\bar{\mathcal{Y}}^q_v\mathcal{Y}^q_v)g^q_v(b)^{-1}p(x))]\!] \ = \\
&g^q_v(b)^{-1}(\bar{\mathcal{Y}}\mathcal{Y})(q^\rho\delta^{-b})[\![p(x)]\!] \ = \\
&\prod_{a\in R^\vee_+,\nu_a\in v}(q_a^{-1}y_a(q^{-\rho/2}\delta^{+b/2}) - y_a(q^{+\rho/2}\delta^{-b/2}))[\![p(x)]\!].
\end{aligned}\tag{3.7}
$$

Finally, $d^q_v \stackrel{def}{=} \pm[\![\mathcal{Y}^q_v\mathcal{X}^q_v]\!]$ can be determined from (3.7) and the relation $1 = p' \ = \ g^q_v(b)^{-1}G^q_v(p^q_b)$ for $b = -r_v$, where p_{-r_v} coincides with the monomial function m_{-r_v} (it follows directly from the definition):

$$
d^q_v \ = \prod_{a\in R^\vee_+,\nu_a\in v}(q_a^{-1}y_a((q\delta_v)^{-\rho/2}) - y_a((q\delta_v)^{+\rho/2}))m_{-r_v}(q^{-\rho}).\tag{3.8}
$$

Let us check that

$$
[\![(\bar{\mathcal{Y}}^q_v - l_\epsilon(w_0)\mathcal{Y}^q_v)(\mathcal{X}^q_v p'(x))]\!] = 0 \ \text{ for } \quad \text{any } p' \in \mathbf{C}[x],
$$

where $l_\epsilon(w_0) = \prod_\nu \epsilon_\nu^{l_\nu(w_0)}$,

$$
\epsilon = \{\epsilon_\nu = -1 \ \text{ if } \ \nu\in v, \ \text{otherwise } \epsilon_\nu = 1\}, \ \nu\in\nu_R.
$$

Following formula (4.18),[C2] we introduce the *q-symmetrizers*, setting

$$
\begin{aligned}
\mathcal{P}^q_v \;&=\; (\pi^q_v)^{-1}\sum_{w\in W}\prod_{\nu}(\epsilon_\nu q_\nu^{1/2})^{\epsilon_\nu(l_\nu(w)-l_\nu(w_0))}T_w,\\
\pi^q_v \;&=\; \sum_{w\in W}\prod_{\nu}(\epsilon_\nu q_\nu^{1/2})^{\epsilon_\nu(2l_\nu(w)-l_\nu(w_0))}.
\end{aligned}
\tag{3.9}
$$

It results from Proposition 3.5 and Corollary 4.7(ibidem) that

$$
\mathcal{P}^q_v(\mathcal{X}^q_v p') = \mathcal{X}^q_v p',\ \hat{\mathcal{P}}^q_v\mathcal{P}^{q=0}_v = \hat{\mathcal{P}}^q_v,\ \text{ if }\ q_\nu = 1\ \text{ for }\ \nu\notin\nu_R.
$$

Hence

$$
\begin{aligned}
&[\![(\bar{\mathcal{Y}}^q_v - l_\epsilon(w_0)\mathcal{Y}^q_v)(\mathcal{X}^q_v p'(x))]\!] = [\![(\bar{\mathcal{Y}}^q_v - l_\epsilon(w_0)\mathcal{Y}^q_v)\mathcal{P}^q_v(\mathcal{X}^q_v p'(x))]\!] =\\
&[\![(\mathcal{Y}^q_v\bar{p}'(Y))\mathcal{P}^q_v(\bar{\mathcal{X}}^q_v - l_\epsilon(w_0)\mathcal{X}^q_v)]\!] = [\![(\mathcal{Y}^q_v\bar{p}'(Y))\{\hat{\mathcal{P}}^q_v\mathcal{P}^{q=0}_v(\bar{\mathcal{X}}^q_v - l_\epsilon(w_0)\mathcal{X}^q_v)\}]\!].
\end{aligned}
$$

The latter equals zero. □

Let us take any set $k = \{k_{\nu_1}\ge k_{\nu_2}\}\in \mathbf{Z}_+$ and put

$$
q(k) = \{\delta^{2k_\nu/\nu}\},\ k\cdot r = \sum_\nu k_\nu r_\nu,\ p^{(k)}_b = p^{q(k)}_b.
\tag{3.10}
$$

The remaining part of the calculation is based on the following chain of the shift operators that will be applied to $p^{(0)}_{b-k\cdot r} = m_{b-k\cdot r}$ one after another:

$$
G^{(k-1)}_{\nu_R}G^{(k-2)}_{\nu_R}\cdots G^{(k-s)}_{\nu_R}G^{(k-s-e)}_{\nu_1}\cdots G^{(0)}_{\nu_1},
\tag{3.11}
$$

where $k_{\nu_1} = s+t,\ k_{\nu_2} = s,\ e = \{e_\nu\},\ e_{\nu_1} = 1,\ e_{\nu_2} = 0,\ k-s = te$, the set $\{1,1\}$ is denoted by 1.

Lemma 3.4 gives that for a certain $D^{(k)}$ (which does not depend on b):

$$
\begin{aligned}
D^{(k)}p^{(k)}_b(q(k)^{-\rho}) \;&=\; m_{b-k\cdot r}(1)\prod_{a\in R^\vee_+,\nu_a\in v(i)}^{0\le i<s+t}\\
&\Big(q(i)_a\, y_a(q(i)^{\rho/2}\delta^{-b(i)/2}) - y_a(q(i)^{-\rho/2}\delta^{b(i)/2})\Big),
\end{aligned}
\tag{3.12}
$$

where $q(i) = q(k(i)),\ b(i) = b-(k-k(i))\cdot r$,

$$
k(i) = ie,\ v(i) = \nu_1\ \text{ if }\ i<t,\ k(i) = i-t+te,\ v(i) = \nu_R\ \text{ if }\ i\ge t.
$$

As to $D^{(k)}$, it equals the right hand side of (3.12) when $b=0$. We note that

$$
q(i)_a = \delta^{2j/\nu_a}\ \text{ for }\ j = k_a+i-s-t,\ k_a = k_{\nu_a},
$$

because $i\ge t$ if $\nu_a\ne\nu_1$ (and $0\le j<k_a$). The relation $(2/\nu)\rho_\nu = r_\nu$ leads to the formulas:

$$
\begin{aligned}
q(i)^{\rho/2}\delta^{-b(i)/2} \;&=\; \delta^{(k(i)\cdot r-b+(k-k(i))\cdot r)/2} \;=\; \delta^{k\cdot r-b/2},\\
y_a(q(i)^{\rho/2}\delta^{-b(i)/2}) \;&=\; \delta^{(k\cdot r-b,a)/2}.
\end{aligned}
\tag{3.13}
$$

Finally, we arrive at the following theorem:

Evaluation Theorem 3.4.

$$p_b^{(k)}(q(k)^{-\rho}) = \frac{m_{b-k\cdot r}(1)}{m_{-k\cdot r}(1)} \prod_{\alpha\in R_+, 0\le j<k_\alpha} \Big(\frac{\delta^{\{(k\cdot r-b,\alpha)+j\}/\nu_\alpha} - \delta^{-\{(k\cdot r-b,\alpha)+j\}/\nu_\alpha}}{\delta^{\{(k\cdot r,\alpha)+j\}/\nu_\alpha} - \delta^{-\{(k\cdot r,\alpha)+j\}/\nu_\alpha}}\Big). \tag{3.14}$$

□

We note that $m_{b-k\cdot r}(1)/m_{-k\cdot r}(1) = |W(b-k\cdot r)|/|W(k\cdot r)| \in \mathbf{Z}_+$. It equals 1 for all $b\in B_-$ when $\prod_\nu k_\nu \neq 0$. Assuming this we have:

$$p_b^{q(k)}(q(k)^{-\rho}) \ = \ \delta^{(k\cdot r,b)} \prod_{\alpha\in R_+, 0\le j<\infty} \Big(\frac{(1-\delta^{2\{(k\cdot r-b,\alpha)+j\}/\nu_\alpha})(1-q_\alpha(k)\delta^{2\{(k\cdot r,\alpha)+j\}/\nu_\alpha})}{(1-q_\alpha(k)\delta^{2\{(k\cdot r-b,\alpha)+j\}/\nu_\alpha})(1-\delta^{2\{(k\cdot r,\alpha)+j\}/\nu_\alpha})}\Big). \tag{3.15}$$

The limit of (3.15) as one of the k_ν approaches zero exists and coincides with (3.14). Since both sides of this formula are rational functions in $q(k)$ and δ we get (0.6) (cf. Theorem 2.4).

We note that actually this paper does not depend very much on the definition of the Macdonald polynomials from the Introduction. We can eliminate μ introducing these polynomials as the eigenfunctions of the L-operators (formula (2.16)). Therefore it is likely that paper [C4] can be extended to give a "difference-elliptic" Weyl dimension formula.

Schwartz functions. In conclusion we will use Macdonald's polynomials to construct pairwise orthogonal functions with respect to the pairing $[\![\ ,\ ]\!]$ in the case of A_n. At the moment, the extension of this construction to other root systems is not known. We will start with the following observation:

Proposition 3.5. *Adding proper roots of δ, the following maps are automorphisms of $\mathfrak{H}$ of type A_n:*

$$\begin{aligned} &\tau:\ X_i\to X_i,\quad Y_i\to X_iY_i\delta^{-c_i},\quad T_i\to T_i,\\ &\omega:\ X_i\to Y_i,\quad Y_i\to Y_i^{-1}X_i^{-1}Y_i\delta^{2c_i},\quad T_i\to T_i,\\ &q_\nu\to q_\nu,\ \delta\to\delta,\ c_i=(b_i,b_i)=i(n-i+1)/(2(n+1)). \end{aligned} \tag{3.16}$$

Proof. Here $b_i=\omega_i$ in the notations from [B]. The proof can be deduced from the topological interpretation of $\mathfrak{H}$ from [C1] in terms of the elliptic braid groups. In the case of A_n the latter group (due to Birman and Scott) is especially simple. Actually these automorphisms are related to the standard generators of $SL_2(\mathbf{Z})$. Let us give another description of τ (as for ω, it can be expressed in terms of τ and φ).

Setting $x_b = \delta^{z_b},\ z_{a+b} = z_a + z_b,\ z_i = z_{b_i},\ a(z_b) = z_b - (a,b),\ a,b \in \mathbf{R}^n$, we introduce the *Gaussian function* $\gamma\ =\ \delta^{\Sigma_{i=1}^n z_i z_{\alpha_i}/2}$, which is considered as a formal series in $x, \log\delta$ and satisfies the following difference relations:

$$\begin{aligned} b_j(\gamma)\ &=\ \delta^{(1/2)\Sigma_{i=1}^n (z_i-(b_j,b_i))(z_{\alpha_i}-\delta_i^j)}\ = \\ \gamma\delta^{-z_j+(b_j,b_j)/2}\ &=\ x_j^{-1}\gamma\delta^{(b_j,b_j)/2}\ \text{ for }\ 1\le j\le n. \end{aligned} \tag{3.17}$$

The Gaussian function commutes with T_j, for $1 \le j \le n$ because it is W-invariant. Since all b_j are minuscule, we use directly formulas (1.12, 2.5) to check that

$$\gamma(X)Y_j\gamma(X)^{-1}\ =\ X_j\delta^{-(b_j,b_j)/2}Y_j\ =\ \tau(Y_j).$$

$\square$

Actually, we can take here an arbitrary W-invariant polynomial g of $\{z_1,\dots,z_n\}$ such that $b(\delta^g)\delta^{-g}$ belong to $\mathbf{C}_\delta[X]$.

We claim that the Schwartz functions $\{\gamma p_b,\ b \in B_-\}$ defined for the Macdonald polynomials $\{p_b\}$ are pairwise orthogonal with respect to the Fourier pairing $[\![\ ]\!]$. Here one should complete $\mathfrak{H}$. Avoiding this we will reformulate the statement as follows:

PROPOSITION 3.6. *The operators $\mathcal{L}_f^\gamma \overset{def}{=} \gamma\mathcal{L}_f\gamma^{-1}$ defined for $f \in \mathbf{C}[y]^W$ are W-invariant (see Theorem 2.3). Moreover, $\varphi(\mathcal{L}_f^\gamma) = \mathcal{L}_f^\gamma,\ \mathcal{L}_f^\gamma(\gamma p_b) = f(q^\rho\delta^{-b})(\gamma p_b)$, and the corresponding eigenvalues (for all f) distinguish different γp_b.*

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